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Note on Prof. Sylvester's "Constructive Theory of Partitions."

By Morgan Jenkins, M. A.

After reading Professor Sylvester's exposition (Act III, Vol. V, No. 3, p. 286, and No. 4 to p. 296) of the mode of using the bends of a regularized graph for the construction of combined partitions, I thought it would be useful to show how to examine the bends of a graph, having given large numbers as elements without the inconvenience of actually constructing the graph. In pursuing this course one additional result has been obtained giving a set of progressions having the common difference two, instead of a set of sequences.

Let $$m_1 a_{(1)} + m_2 a_{(2)} + \dots \qquad + m_j a_{(j)} = P$$

and $$\mu_1 \alpha_{(1)} + \mu_2 \alpha_{(2)} + \dots \qquad + \mu_j \alpha_{(j)} = P$$

represent two conjugate partitions of a number P; $a_{(1)}, a_{(2)} \dots a_{(j)}$ being the j different elements of one partition, arranged in descending order of magnitude; $m_1, m_2 \dots$ et cetera, the numbers of times those elements are taken; and let $a_1, a_2, a_3, \dots$ et cetera, represent the elements of the same partition, taken in order, counting repetitions; that is $a_1, a_2, \dots a_{m_1}$, each equal to $a_{(1)}$, and so on; also let the μ's and the α's represent corresponding quantities in the conjugate partition. Then we have

$$\alpha_{(1)} = \Sigma m_j = m_1 + m_2 + \dots + m_j;\ \alpha_{(2)} = \Sigma m_{j-1};\ \dots\ \alpha_{(j)} = m_1,$$

$$u_1 = a_{(j)};\ \mu_2 = a_{(j-1)} - a_{(j)};\ \mu_3 = a_{(j-2)} - a_{(j-1)}\ \dots\ \mu_j = a_{(1)} - a_{(2)},$$

and reciprocally $\quad a_{(1)} = \Sigma \mu_j;\quad m_1 = \alpha_{(j)}$, and so on.

The two partitions are of the same extent j, as well as of the same content P; and any process which alters the extent of one partition will alter in like manner the extent of the conjugate partition.

The selection of suitable examples will be facilitated if we notice that, instead of supposing the coefficients and elements of one partition to be given, we may take any j different numbers as the elements of one partition, and any other j different numbers (not necessarily differing from the first set) as the elements of the second: then the coefficients of either partition are equal to the differences of consecutive elements of the other partition, the coefficient of the highest element in one partition being equal to the lowest element in the other.

The content of each partition is

$$a_{(j)}\alpha_{(1)} + [a_{(j-1)} - a_j]\,\alpha_{(2)} + [a_{(j-2)} - a_{(j-1)}]\,\alpha_{(3)} + \ldots [a_{(1)} - a_{(2)}]\,\alpha_{(j)}$$

which equals

$$\{a_{(j)}\alpha_{(1)} + a_{(j-1)}\alpha_{(2)} + \ldots + a_{(1)}\alpha_{(j)}\} - \{a_{(j)}\alpha_{(2)} + a_{(j-1)}\alpha_{(3)} + \ldots a_{(2)}\alpha_{(j)}\}$$

a result which is unaltered when the a's and α's are interchanged.

In a regularized graph it will be seen that the different elements are th coordinates of the 'out corners' of the outline of the graph: thus at T, in Fig. 1 the number of nodes in the row and column passing through T may be denote by $a_{(f)}$ and $\alpha_{(j+1-f)}$ respectively, the sum of the subscripts being $j+1$. At a 'in-corner,' as at S, the number of nodes in the row and column passing through S, reckoned up to and including S, but not beyond, are the same as at the adjacent out-corners R and T respectively: hence the coordinates of an in-corner may be taken to be $a_{(f+1)}$ and $\alpha_{(j+1-f)}$, the sum of the subscripts being $j+2$ instead of $j+1$.

The diagonal of the Durfee-square terminates either at an 'out-corner' (Fig. 1) or at an 'in-corner' (Fig. 2), or it divides a column (Fig. 3) or a row (Fig. 4). If we use i to denote the number of bends in the graph, in each of the four cases i is equal to one of the elements of the graph, either $a_{(h)}$ or $\alpha_{(h)}$.

FIG. 1.

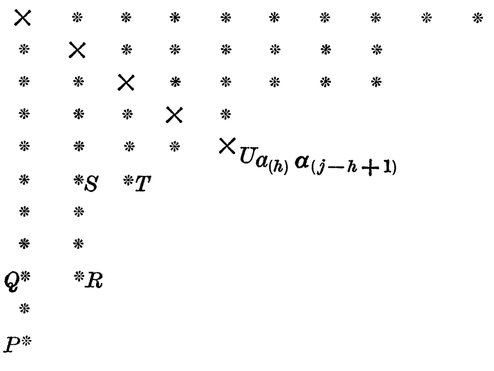


FIG. 2.

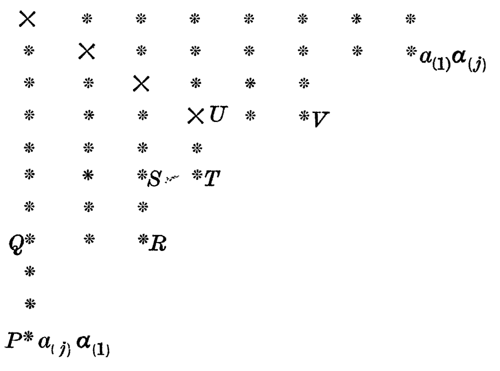


FIG. 3.

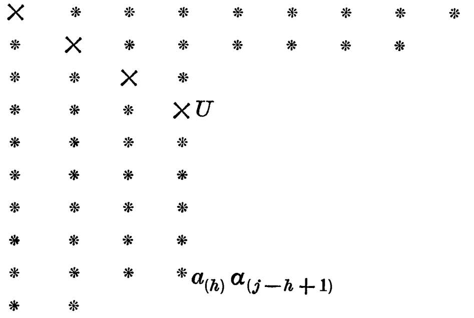


FIG. 4.

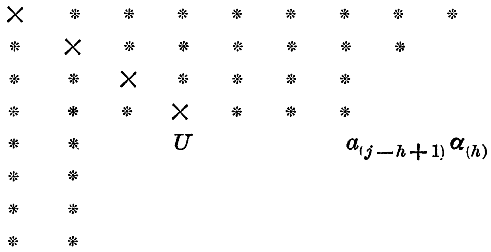
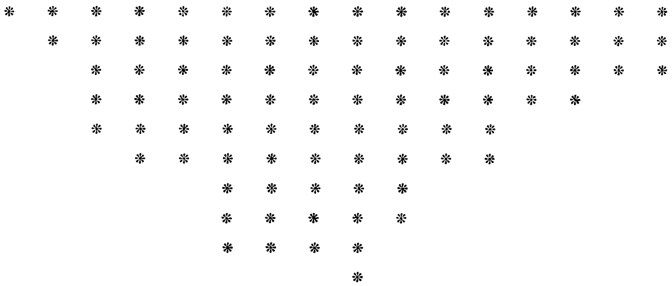


FIG. 5.



In Fig. 1 $a_{(h)} = \alpha_{(j-h+1)} = i$; $a_i = \alpha_i = i$.
In Fig. 2 $a_{(h)} = \alpha_{(j-h+2)} = i$,
$a_i = a_{(h-1)}$, which is greater than i,
$\alpha_i = \alpha_{(j-h+1)}$, which is greater than i.
In Fig. 3 $a_{(h)}$ is less than $\alpha_{(j+h+1)}$ and greater than $\alpha_{(j-h+2)}$; $a_i = a_{(h)} = i$,
$\alpha_i = \alpha_{(j-h+1)}$ which is greater than i.
In Fig. 4 $\alpha_{(h)}$ is less than $a_{(j-h+1)}$ and greater than $a_{(j-h+2)}$
$a_i = a_{(j-h+1)}$, which is greater than i,
$\alpha_i = \alpha_{(h)} = i$.

Thus in every case we can find an element of one partition, which is between or not outside the pair of consecutive elements of the other partition, which is formed of the corresponding element and the one next below it; and the first named element gives the number of bends in the graph which represents the partition.

Let us now consider the effect of adding two regularized graphs together, row to row, in order. If we suppose them to be placed side by side (Fig. 5) and then all the nodes in every row to be pushed close to the left-hand side, the same effect is produced as if the columns were picked out and rearranged in order of magnitude. The number of different columns in the regularized graph so formed is unaltered by the rearrangement.

Hence if j be the extent, that is the number of different elements in either set of conjugate elements, of one partition, the conjugate elements being denoted by a's and α's, and k, b and β be corresponding letters for another partition, then the extent of the combined partition $a_1 + b_1$, $a_2 + b_2$, et cetera, is equal to the number of different elements in $\alpha_{(1)}, \alpha_{(2)}, \ldots \alpha_{(j)}, \beta_{(1)}, \beta_{(2)} \ldots \beta_{(k)}$, considered as a single series, that is, if l be the number of α elements which are equal to β elements, the extent of the combined partition is $j + k - l$. This theorem may be applied to the bends of a graph. The numbers of nodes in these bends are

$$a_1 + \alpha_1 - 1,\ a_2 + \alpha_2 - 3,\ a_3 + \alpha_3 - 5 \ldots a_i + \alpha_i - (2i - 1).$$

If we omit the negative terms we have a regularized graph which is the sum of two regularized graphs, and which exceeds the original graph in content by $1 + 3 + 5 + \ldots + (2i - 1)$ or by i^2, which is the content of the Durfee-square. The second of the two partial graphs may be supposed to be turned through a right angle, so that α_1 is in a line with a_1, α_2 with a_2, ... and α_i with a_i. The extent of the new graph, that is the number of different elements in either set of its conjugate elements will be found, in every case, to be $j - g$, where g is the number of elements in either set of elements of the original graph which are less than the corresponding conjugate element, but equal to some other conjugate element.

Thus in the combined graph formed by the addition of the parts of graph (Fig. 1), if we cut off all the rows below the Durfee-square, the different columns in the remaining graph are $\alpha_{(j)}, \alpha_{(j-1)} \ldots \alpha_{(j-h+1)}$: if we cut off all the columns to the right of the Durfee-square and turn the remaining graph round through a right angle, the different columns in this remaining graph are now $a_{(j)}, a_{(j-1)} \ldots a_{(h)}$. In these two series $a_{(h)} = \alpha_{(j-h+1)}$, and is not included in the g equalities between the elements of the original graph, the remaining equalities are g in number according to the above-stated definition of g. Therefore the number of elements in the two series, including repetitions, is $h + (j - h + 1)$ or $j + 1$, and the number excluding repetitions is $(j + 1) - (g + 1)$ or $j - g$. In the case of Fig. 2, the columns of one graph are $\alpha_{(j)}, \alpha_{(j-1)} \ldots \alpha_{(j-h+2)}$, and the columns, formed out of rows, of the other graph are $a_{(j)}, a_{(j-1)} \ldots a_{(h)}$; $a_{(h)}$ being equal to $\alpha_{(j-h+2)}$ and less than $\alpha_{(j-h+1)}$ is reckoned in the g equalities. Therefore the number of elements in the two series is, including repetitions, $(h-1)+(j-h+1)$ or j, and is, excluding repetitions, $j-g$. In the case of Fig. 3, the columns of one graph are $\alpha_{(j)}$, $\alpha_{(j-1)} \ldots \alpha_{(j-h+2)}\, a_{(h)}$, $a_{(h)}$ taking the place of an α element at the end in consequence of the diagonal of the Durfee-square falling between two corners of the original graph; the columns, formed out of rows, of the other partial graph are $a_{(j)}$, $a_{(j-1)} \ldots a_{(h)}$. $a_{(h)} = a_{(h)}$ is to be counted in the equalities of these two series, and is not one of the g equalities of the original graph. Therefore the number of elements in the two series is, including repetitions, $j+1$, and is, excluding repetitions, $(j+1)-(g+1)$ or $j-g$. The case of Fig. 4 is exactly like that of Fig. 3, interchanging a's and α's.

If we subtract the terms of a single ascending progression having the common difference r from a constant quantity, the remainders will form a single descending progression having the common difference r. If for the constant quantity we substitute the terms of a descending series, containing repetitions, every break in the repetitions produces a break in the progressions arising from the subtraction. Hence the number of distinct progressions produced must be the same as the number of different numbers in the series from which we subtract; also the last number of one progression must exceed the first number of the next progression by a number greater than r.

In applying the theorem with regard to the subtraction of the terms of a single progression from the terms of another series containing repetitions, we should have to notice whether any of the remainders were zero or negative. In this case all the remainders are positive, because they represent the bends of the original graph, and the last term of the series is not zero.

It follows that the numbers of nodes in the bends of a regularized graph when placed in descending order of magnitude contain $j-g$ distinct progressions, having the common difference 2, where j is the number of different elements in either set of conjugate elements of the graph, and g is the number of elements in either set which are less than their corresponding element but equal to some other element of the conjugate set: also the excess of the last number of one progression over the first number of the next progression is greater than two.

EXAMPLE 1. Let

$$\begin{array}{cccccc} 11 & 10 & 7 & 5 & 3 & 2 \\ 4 & 5 & 7 & 9 & 12 & 13 \end{array}$$

be the elements of two conjugate partitions written one underneath the other in contrary order of magnitude. The extent (j) is equal to 6, the content is equal to 89, and the coefficients are as given by the equations

$$4.11 + 1.10 + 2.7 + 2.5 + 3.3 + 1.2 = 89$$
$$2.13 + 1.12 + 2.9 + 2.7 + 3.5 + 1.4 = 89$$

The numbers of nodes in the bends of the graph which represents these conjugate partitions are found as follows:

1 is less than 4 in lower line, so we take 11 from upper line,
" " 2 " " " " 13 " " "

and the number of nodes in the first bend is $11 + 13 - 1$.

2 is less than 4 in lower line, so we take 11 from upper line,
2 is equal to 2 " " " 13 " " "

and the number of nodes in the second bend is $11 + 13 - 3$.

3 is less than 4 in lower line, so we take 11 from upper line,
3 is equal to 3 " " " " 12 " " "

and the number of nodes in the third bend is $11 + 12 - 5$, and so on. Thus we have for the bends

$$\begin{array}{r} 11 + 13 - 1 = 23 \\ 11 + 13 - 3 = 21 \\ 11 + 12 - 5 = 18 \\ 11 + 9 - 7 = 13 \\ 10 + 9 - 9 = 10 \\ 7 + 7 - 11 = 3 \\ 7 + 7 - 13 = 1 \\ \hline 89 \end{array}$$

Since two corresponding elements 7 are equal to each other, 7 is the number of bends. There is one pair of non-corresponding equal elements which do not exceed 7, viz. 5, 5; therefore $g = 1$; $j = 6$ and $6 - 1$ or 5 is the number of distinct progressions having the common difference 2. The number of elements which appear in the formation of the bends are $j + 1$ or 7, viz. 11, 10 and 7 in one set and 13, 12, 9, 7 in the other set.

In the next example only the distinctive points will be noticed.

EXAMPLE 2.

$$\begin{array}{ccccccc} 16 & 13 & 12 & 9 & 4 & 3 & 1 \\ 2 & 4 & 9 & 11 & 13 & 14 & 16 \end{array}$$

$$2.16 + 2.13 + 5.12 + 2.9 + 2.4 + 1.3 + 2.1 = 149$$
$$1.16 + 2.14 + 1.13 + 5.11 + 3.9 + 1.4 + 3.2 = 149$$

$j = 7$. Two conjugate elements, 9, which are only one place removed from each other are equal, therefore 9 is the number of bends. The number of pairs of non-corresponding equal elements which do not exceed 9 is 2, viz. 4, 4 and 9, 9. Therefore $g = 2$.

BENDS.

$$\begin{array}{r} 16 + 16 - 1 = 31 \\ 16 + 14 - 3 = 27 \\ 13 + 14 - 5 = 22 \\ 13 + 13 - 7 = 19 \\ 12 + 11 - 9 = 14 \\ 12 + 11 - 11 = 12 \\ 12 + 11 - 13 = 10 \\ 12 + 11 - 15 = 8 \\ 12 + 11 - 17 = 6 \\ \hline 149 \end{array}$$

Also $7-2$ or 5 is the number of distinct progressions having the common difference 2.

In this case the number of elements from the two sets appearing in the formation of the bends is j or 7 and not $j+1$, viz. 16, 13, 12 from one set and 16, 14, 13, 11 from the other set.

EXAMPLE 3.

$$\begin{array}{ccccccc} 13 & 12 & 11 & 6 & 5 & 3 & 2 \\ 2 & 3 & 5 & 7 & 8 & 9 & 10 \end{array}$$

$$2.13+1.12+2.11+2.6+1.5+1.3+1.2=82$$

$$2.10+1.9+2.8+1.7+5.5+1.3+1.2=82$$

$j=7$; $g=3$ from the equal pairs 2, 2; 3, 3 and 5, 5.

BENDS.

$$\begin{aligned} 13+10-1&=22\\ 13+10-3&=\underline{20}\\ 12+9-5&=\underline{16}\\ 11+8-7&=12\\ 11+8-9&=\underline{10}\\ 6+7-11&=\underline{2}\\ &\,\underline{\underline{82}} \end{aligned}$$

EXAMPLE 4.

$$\begin{array}{ccc} 12 & 10 & 9 \\ 2 & 5 & 6 \end{array}$$

$$2.12+3.10+1.9=63$$

$$9.6+1.5+2.2=63$$

$j=3$; $g=0$. Number of bends $=6$, because 6 is between the corresponding element 9 and 0. $j-g=3$, and there are 3 distinct progressions.

BENDS.

$$\begin{aligned} 12+6-1&=17\\ 12+6-3&=\underline{15}\\ 10+6-5&=11\\ 10+6-7&=9\\ 10+6-9&=\underline{7}\\ 9+6-11&=\underline{4}\\ &\,\underline{\underline{63}} \end{aligned}$$

The method here applied to the bends of the original graph may be used to establish the theorem enunciated and proved by Professor Sylvester, namely, that if we interpolate with the bends of the original graph the bends of the graph obtained by cutting off the highest column from the original graph, the series so obtained will contain j sequences.

The series is $\alpha_1+\alpha_1-1$, $\alpha_1+\alpha_2-2$, $\alpha_2+\alpha_2-3$, $\alpha_2+\alpha_3-4$, . . . $\alpha_i+\alpha_i-(2i-1)$, α_i-i.

The last term is α_i-i (as stated in Vol. V, Number 3, p. 288), and it may be written α_i+i-2i. If we omit the negative terms we have the sum of two partial graphs which are formed thus: Double the number of terms of every element from α_1 to α_i inclusive; also double the number of terms of every element from α_1 to α_i inclusive, but cut off the first element α_1 from the beginning and attach an element i at the end.

For each of the three cases (Fig. 1, Fig. 2 and Fig. 4) where i is equal to an α element, say $\alpha_{(h)}$, the conjugate elements, whose differences give the number of times $a_{(1)}$, $a_{(2)}$, et cetera, are taken, are $2\alpha_{(j)}$, $2\alpha_{(j-1)} \ldots 2\alpha_{(h)}$, all even numbers.

The conjugate elements whose differences give the number of times $\alpha_{(1)}$, $\alpha_{(2)}$ $\ldots$, et cetera, are taken, are $2a_{(j)} - 1$, $2a_{(j-1)} - 1$, $\ldots$ $2a_{(j-h+2)} - 1$, $2\alpha_{(h)}$, all odd numbers except the last. The cutting off of the first element to form the second partial graph explains why all the conjugate elements but the last are odd: the reason why the last is even is that one element, i, has been added to the $2\{\alpha_{(h)} - a_{(j-h+2)}\}$ elements, each equal to $\alpha_{(h)}$; and i being equal to $\alpha_{(h)}$ in the three cases of Fig. 1, Fig. 2 and Fig. 4, the conjugate element becomes $2\alpha_{(h)}$ instead of $2\alpha_{(h)} - 1$. The number of different elements in the two sets of conjugate elements taken as a single series is $j - h + 1 + h - 1$ or j, the element $2\alpha_{(h)}$ which is the only element common to the two series being only counted once. When we subtract the terms of the single sequence $1, 2, 3, \ldots 2i$, we shall have the remainders forming j distinct sequences. In the case of Fig. 1, and Example 1, the last term is 0, $a_i - i$ being equal to 0; but it can be shown that this 0 is preceded by 1, α_i being also equal to i in this case. Therefore if we cut off this 0 we shall have j sequences left. If we examine the case of Fig. 3 and Example 3, we shall find there are $j + 1$ sequences, including the final term, which is 0; but the penultimate term must be greater than 1.

Hence if we cut off the final zero term we have j sequences left.